First passage of a particle in a potential under stochastic resetting: A vanishing transition of optimal resetting rate

Saeed Ahmad,\textsuperscript{1} Indrani Nayak,\textsuperscript{1} Ajay Bansal, Amitabha Nandi, and Dibyendu Das

Physics Department, Indian Institute of Technology Bombay, Mumbai 400076, India

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First passage in a stochastic process may be influenced by the presence of an external confining potential, as well as “stochastic resetting” in which the process is repeatedly reset back to its initial position. Here, we study the interplay between these two strategies, for a diffusing particle in a one-dimensional trapping potential $V(x)$, being randomly reset at a constant rate $r$. Stochastic resetting has been of great interest as it is known to provide an “optimal rate” ($r_*$) at which the mean first passage time is a minimum. On the other hand, an attractive potential also assists in the first capture process. Interestingly, we find that for a sufficiently strong external potential, the advantageous optimal resetting rate vanishes (i.e., $r_* \to 0$). We derive a condition for this optimal resetting rate vanishing transition, which is continuous. We study this problem for various functional forms of $V(x)$, some analytically, and the rest numerically. We find that the optimal rate $r_*$ vanishes with a deviation from the critical strength of the potential as a power law with an exponent $\beta$ which appears to be universal.

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I. INTRODUCTION

Survival and first passage problems are of great interest in stochastic process literature [1–3]. Such questions have been studied in the theory of random walks [4–6], polymer and interface kinetics [7–9], chemical reactions [10,11], diffusion in quenched flow fields [12,13], algorithmic problems [14], and biophysics [15–17]. In the course of the stochastic evolution of single- or multiparticle systems, first passage is said to occur when an event of crucial interest happens for the first time. The distribution of timescales of the first occurrence of the event, as well as various cumulants of the distribution are of interest [18]. The mean first passage time (MFPT) is typically infinite for simple diffusive processes in open geometry, but finite in the case of closed geometries and in the presence of spatially attractive potentials.

Recently, stochastic resetting (SR) in stochastic processes has become a topic of active research [19–37]. In SR problems, a stochastic process is repeatedly returned to its initial position after random time intervals. This ensures that the process does not drift off very far from the initial position, and as a result a (nonequilibrium) steady state is attained at large times. In addition to this, the original stochastic process may attempt a first passage event. A natural question is whether SR assists or impedes the process of first capture. For simple diffusion with SR at a constant rate, Evans and Majumdar [19] showed that the MFPT becomes finite—thus SR assists in first capture. Moreover, there is an optimal resetting rate (ORR) at which the MFPT is a minimum. Since then, this phenomenon has been studied in a variety of different scenarios with different rules for resetting. The resetting time can be completely deterministic or taken from a power-law distribution [21,24]. Similarly, the rate may have an explicit time [22,23] or position [27] dependence. The optimality of such resetting processes has been studied for multiple walkers [24,25], and fluctuating interfaces [31,32]. For any process with a constant resetting rate, it has been shown that at ORR, variance of the first passage times is equal to the square of MFPT [34]. Furthermore, for more general SR time distributions, many universal identities and inequalities have been derived [35].

We recall that for simple diffusion in one dimension, SR at a constant rate produces an advantage, and an ORR exists where MFPT is minimum. Can this advantage be nullified? A simple way to do that is by introducing an additional attractive potential $V(x) = kx^n$, with $k > 0$ and $n \in (0, \infty)$, which drives the particle towards the capture site at $x = 0$. There exist earlier studies of SR in the presence of diverse external potentials [28–30]. In the absence of a potential (i.e., for $n = 0$), SR is advantageous. Similarly for any $n > 0$, if the strength of the potential $k \to 0$, it is as good as a flat potential—hence SR helps towards first passage as in the $n = 0$ case. On the other hand, if $k \to \infty$, the particle would be driven to the origin by an enormous advective force and first passage would happen instantly—no amount of resetting or any other strategy can make the first passage time any lower. But for any finite $k$, it remains an open question whether SR would still be a helpful strategy towards a speedy first passage. In this paper, we explore how the potential competes with SR for dominance, and beyond a critical threshold strength, renders SR to be redundant. We find that by tuning and increasing the strength $k$, ORR can be made to vanish for $k$ greater than a threshold value $k_c(n)$. Thus in the presence of a sufficiently strong attractive potential, SR no longer helps in first capture. In this paper, we study this ORR vanishing transition, and find a universal behavior in its vicinity—ORR scales as $\sim[k_c(n) - k]^\beta$ for $k < k_c$, with $\beta = 1$ (independent of $n$). Following Ref. [11], we also study the transition with

\textsuperscript{1}saeedmalik@iitb.ac.in
\textsuperscript{1}inayak21@phy.iitb.ac.in
reset followed by a stochastic time overhead (with a mean time \((T_{\text{on}})\), as would be expected in a Michaelis-Menten reaction scheme (MMRS).

The idea of an ORR vanishing transition is not entirely new [11,29]. Unlike our paper, where we vary the strength of the potential, in Ref. [11], an ORR vanishing transition was discussed by varying \((T_{\text{on}})\). The special case of \(n = 1\) has been independently studied in another recent work [38].

In Sec. II, we define the problem mathematically and derive the condition which solves for the transition point \(k_c(n)\). Then we argue why a universal exponent \(\beta = 1\) is expected. In Sec. III, we demonstrate these facts through exact results for linear \((n = 1)\), harmonic \((n = 2)\), and box \((n \to \infty)\) potentials. In Sec. IV, we extend some of the results to the case of reset followed by a refractory period. In Sec. V, we present a numerical scheme to study the problem, and apply it to cases of the cubic \((n = 3)\) and quartic \((n = 4)\) potentials, as well as a potential which is a nonmonotonic function of \(x\). We provide concluding remarks in Sec. VI.

II. THE PROBLEM AND SOME GENERAL RESULTS

We consider a diffusing particle (with diffusion constant \(D\)) initially at \(x = x_0\), subject to an external attractive potential \(V(x) = kx^n\) with \(k > 0\) and \(n \in (0, \infty)\). There is an absorbing boundary at \(x = 0\). Additionally, the particle position is stochastically reset back to \(x_0\) at a constant rate \(r\). We are interested in the first passage of the particle as a result of the interplay of SR and the potential.

As is often done in first passage problems [18], one may start with the backward differential Chapman-Kolmogorov equation for the probability \(Q(x,t)\) of the particle to survive until time \(t\), starting from any initial position \(x\),

\[
\frac{\partial Q}{\partial t} = D \frac{\partial^2 Q}{\partial x^2} - V(x) \frac{\partial Q}{\partial x} - rQ + rQ_0, \tag{1}
\]

where \(Q \equiv Q(x,t)\), and \(Q_0 \equiv Q(x_0,t)\). Note that for our problem, the spatial derivative of the potential \(V(x) = kx^n\) in the above equation. The initial condition is \(Q(x,0) = 1\), and the absorbing boundary condition is \(Q(0,t) = 0\). Note that \(Q\) is finite, while its spatial derivatives vanish as \(x \to \infty\). On finding \(Q(x,t)\), one may replace \(x_0\) (the particular specified initial position) and solve for \(Q(x,t)\).

Taking a Laplace transformation with respect to \(t\) and defining \(q \equiv q(x,s) = \int_0^\infty dtQ(x,t)e^{-st}\), \(q_0 \equiv q(x_0,s)\), and \(y(x,s) = q(x,s) - \frac{q_0}{r}e^{-st}\), from Eq. (1) we get the following equation for the function \(y\),

\[
\frac{d^2 y}{dx^2} - ny^n + \frac{dy}{dx} - \alpha y = 0. \tag{2}
\]

In the above equation, \(\gamma = k/D\) and \(\alpha = \sqrt{r + s/D}\). Equation (2) is not easy to solve for general \(n\), except for some special cases. One general observation can be made by converting the standard form of Eq. (2) to its normal form, \(u'' + f(x)u = 0\), where \(y = ue^{\frac{\alpha}{\gamma}x}\) and \(f(x) = \frac{-\alpha^2 - \frac{\alpha^2}{\gamma^2}x^n}{\gamma^n} = 0\).

Since \(f(x) < 0\) for increasing \(x\), it follows from Sturm’s theorem [39] that \(u\) has at most one zero. From this it follows that \(y\) is a monotonically increasing function of \(x\), between \(y(0,s) = -\frac{\gamma^{n+1}}{(r+1)!}\) and \(y(\infty,s) = 0\) (which follow from initial conditions), without any zero crossing in between.

In any stochastic process with resetting, it has been shown quite generally [34] that the MFPT is

\[
(T_r) = \frac{1 - \tilde{F}(r)}{r\tilde{F}(r)}, \tag{3}
\]

where \(\tilde{F}(s)\) is the Laplace transform of the first passage probability distribution in the corresponding problem “without resetting.” For our problem \(\tilde{F}(x,s) = 1 - s q_1(x,s)\), where \(q_1(x,s) = \lim_{r \to 0} q(x,s)\). Often \(T_r\) has a minimum at an ORR \(r = r_*\), i.e., \(r_* = \arg \min \langle T_r \rangle\). In the current problem, tuning the strength \(k\) of the potential, it may be made to dominate over SR and thus make ORR vanish, i.e., \(r_* \to 0\). Near the latter transition point, since \(r_*\) would be small, one may approximate MFPT in Eq. (3) as a series in \(r\) up to \(O(r^2)\) [see the Appendix for \(O(r^3)\)],

\[
\langle T_r \rangle \approx \langle T \rangle - r \frac{\sigma^2 - \langle T \rangle^2}{2} + r^2 \frac{1}{6} \langle T^3 \rangle - \sigma^2 \langle T \rangle. \tag{4}
\]

Here, the various moments on the right-hand side of Eq. (4) are for first passage times “without resetting”; similar expansions have been studied earlier [11,34]. We note that in the limit of small \(r_*\), the derivative of Eq. (4), i.e., \(d\langle T_r \rangle/dr\) \(r=r_*\) \(= 0\), yields (see the Appendix)

\[
r_* = \frac{1}{4} \left( \frac{\sigma^2 - \langle T \rangle^2}{\langle T^3 \rangle - \sigma^2 \langle T \rangle} \right). \tag{5}
\]

This would imply two things. First, the ORR vanishing transition \((r_* \to 0)\) coincides with the condition

\[
\sigma^2 = \langle T \rangle^2. \tag{6}
\]

Thus it happens when without resetting, the variance of first passage times due to the tuned potential attains the same value as the square of the MFPT. This means that the potential strength \(k = k_c(n)\) at which the transition happens may be solved from Eq. (6). Second, if \(r_*\) vanishes continuously, the expression on the right-hand side of Eq. (5) is expected to scale as follows,

\[
r_* \sim [k_c(n) - k]^\beta. \tag{7}
\]

If an analytic Taylor expansion of \(r_*\) exists in \([k_c(n) - k]\) with the first term nonvanishing, we would expect the exponent \(\beta = 1\). We would see below that this appears to be true for all the potentials we consider. In what follows, we will often use dimensionless counterparts of \(k\) and \(r_*\), namely, \(K = (\frac{k}{D})^\frac{1}{D} x_0\) and \(z_*^2 = r_* x_0^2 / D\).

In addition to a stochastic process (which for our case is a random walk in a potential) and SR, in a chemical MMRS, there is typically an inert period after reset, with a mean time \((T_{\text{on}})\). The latter problem has been studied generally in Refs. [10,11,34]. Yet with the aim of deriving a few exact results of our interest, we would note a few relevant formulas from those works. The MFPT is

\[
\langle T_r \rangle = \frac{[r(T_{\text{on}}) + 1 - \tilde{F}(r)]}{r\tilde{F}(r)}, \tag{8}
\]
and expanding this in small \( r \) near the transition [as in Eq. (4)], we may set \( d\langle T_r \rangle/dr|_{r=r_0}=0 \), and obtain (for small \( r_0 \))
\[
 r_0 = \frac{1}{4} \frac{(\sigma^2 - (T)^2 - 2\langle T_{0n} \rangle(T))}{\left(\frac{1}{2}(T^2) - \frac{\sigma^2}{2}(T - (T)^2)\right)}.
\] (9)

The condition to locate the ORR vanishing transition is then revised from Eq. (6) to the following,
\[
\sigma^2 = (T)^2 + 2\langle T_{0n} \rangle(T).
\] (10)

Again, as noted in our discussion below Eq. (7), \( r_0 \sim [k_c(n) - k] \).

We proceed below to study some special values of \( n \) analytically exactly, and a couple of others numerically, to test our expectations in Eqs. (6), (10), and (7).

### III. ANALYTICAL RESULTS FOR ORR TRANSITION

#### A. Linear potential (\( n = 1 \))

Substituting \( n = 1 \) in Eq. (2), we get the following,
\[
d^2y \over dx^2 - \gamma \frac{dy}{dx} - \alpha^2 y = 0,
\] (11)
the general solution for which is
\[
y(x, s) = A(x)e^{\left[\sqrt{\gamma^2 + \alpha^2}x\right]} + B(x)e^{\left[-\sqrt{\gamma^2 + \alpha^2}x\right]}.\]
(12)

The boundary conditions \( y(0, s) = -\frac{q_0 + 1}{r + s} \) and \( y(\infty, s) = 0 \) fix \( A(x) \) and \( B(x) \), and give a solution for \( y(x, s) \) and hence \( q(x, s) \) as follows,
\[
q(x, s) = \frac{q_0 + 1}{r + s}\left[1 - e^{\left[-\sqrt{\gamma^2 + \alpha^2}x\right]}\right].
\]
(13)

Using Eq. (13) or otherwise, without resetting (i.e., \( r = 0 \)), \( \tilde{F}(x_0, s) = \exp\left([\sqrt{\gamma^2 + \alpha^2} - \alpha_0^2]x_0\right) \), where \( \alpha_0 = \sqrt{\beta} \). This leads to \( \langle T \rangle = -\frac{dF}{dx}_{|x=0} = \frac{\alpha_0^2}{\gamma} \) and \( \langle T^2 \rangle = \frac{d^2F}{dx^2}_{|x=0} = \langle T \rangle^2 + 2\alpha_0^2 + 2\gamma \). Then, according to Eq. (6), the ORR vanishing transition happens at a threshold potential strength
\[
k_c = \frac{2D}{x_0}.
\] (14)

Note that arriving at the above result did not require us to refer to the actual problem with resetting, but we may also derive it by starting with the expression for MFPT under SR (i.e., \( r > 0 \)),
\[
\langle T_r \rangle = q_0(x_0, s)|_{s=0} = \frac{1}{r} \left[e^{\left[\sqrt{\gamma^2 + \alpha^2} - \gamma\right]x_0} - 1\right].
\] (15)

As may be seen from Fig. 1(a), the plot of \( \langle T_r \rangle \) vs \( r \) [following Eq. (15)] has a minimum at \( r = r_c(k) \) (ORK) for \( k < k_c \), and for \( k \geq k_c \), ORR is zero. The value of \( r_c \) (for \( k < k_c \)) may be obtained from the condition \( d\langle T_r \rangle/dr|_{r=r_0} = 0 \) which is a transcendental equation as follows,
\[
\frac{r_0}{2\sqrt{\left(\frac{\gamma_0}{r}\right)^2 + \sqrt{\frac{\gamma_0}{r}}}} = 1 - e^{\left[-\sqrt{\gamma^2 + \alpha^2}\right]}.
\]
(16)

In Fig. 2(a), a dimensionless ORR \( z^2 \) is plotted against a dimensionless potential strength \( K = \gamma x_0 = kx_0/D \), following Eq. (16) (see the solid line). We see that the transition is at \( K = K_c = 2 \), i.e., \( k_c = 2D/x_0 \), as we found in Eq. (14). In the vicinity of \( K = K_c \), for \( K \ll K_c \), using the moments \( \langle T \rangle, \langle T^2 \rangle \), and \( \langle T^3 \rangle = -\frac{\alpha_0^6}{\Gamma(\gamma)}\frac{\Gamma(\frac{\alpha_0^2}{\gamma})}{\Gamma(\frac{\alpha_0^2}{\gamma} + 1)} \) in Eq. (5), we get \( r_0 \) and hence
\[
z^2 = \frac{r_0^2}{x_0^2} = \frac{3}{2}(K_c - K).
\]
(17)

The above Eq. (17) may also be obtained from Eq. (16) by expanding in small \( r_0 \) and \( (K_c - K) \). This exact linearized form [plotted in Fig. 2(a) as a dashed line] shows that the exponent \( \beta = 1 \), as expected in Eq. (7).

#### B. Quadratic potential (\( n = 2 \))

For \( n = 2 \), Eq. (2) may be transformed by substituting
\[
x = \sqrt{\frac{2\gamma}{\gamma}}, \quad \gamma(x(\xi)) = w(\xi)
\]
to the familiar confluent hypergeometric equation [39]
\[
\xi \frac{d^2w}{d\xi^2} + (\gamma - \xi) \frac{dw}{d\xi} = aw = 0,
\]
(18)
with \( c = 1/2 \) and \( a = \alpha^2/4\gamma \). The general solution in terms of the confluent hypergeometric function \( F_1(a, c; \xi) \) of the first kind is known. Transforming back to variable \( x \), we have the general solution
\[
y(x) = A_2 \left[F_1\left(\frac{\alpha^2}{4\gamma} - \frac{3}{4}; \frac{3}{2} \gamma x^2\right) + C_2 x^{\frac{3}{2}} F_1\left(\frac{\alpha^2}{4\gamma} + \frac{1}{2}; \frac{3}{2} \gamma x^2\right)\right].
\]
(19)

Using the boundary condition \( y(\infty, s) = 0 \) and the known asymptotic form \( \lim_{x \to 0} F_1(a, c; \xi) = \frac{\Gamma(c)}{\Gamma(a)} e^{x} x^{a-c} \), we get
\[
C_2 = -\frac{\Gamma(\frac{\alpha^2}{4\gamma} + 1)}{\Gamma(\frac{\alpha^2}{4\gamma})}. \quad \text{The boundary condition } y(0, s) = -\frac{q_0 + 1}{r + s} \text{ implies } A_2 = -\frac{q_0 + 1}{r + s}. \quad \text{Putting these together, we have}
\]
\[
q(x, s) = \left(\frac{q_0 + 1}{r + s}\right)\left[1 - G(x, s)\right],
\]
(20)
where
\[
G = \left[F_1\left(\frac{\alpha^2}{4\gamma} - \frac{3}{4}; \frac{3}{2} \gamma x^2\right) \frac{kx^2}{D} \right] -2\left[\frac{\Gamma(\frac{\alpha^2}{4\gamma} + 1)}{\Gamma(\frac{\alpha^2}{4\gamma})}\right] x^{\frac{3}{2}} F_1\left(\frac{\alpha^2}{4\gamma} + \frac{1}{2}; \frac{3}{2} \gamma x^2\right) \frac{kx^2}{D}\right].
\]
(21)

Note that for the problem without resetting, \( \tilde{F}(x, s) = G(x, s)|_{s=0} \). One may proceed to get \( \langle T \rangle \) and \( \langle T^2 \rangle \) from \( \tilde{F}(x, s) \), but since derivatives of the gamma functions and confluent hypergeometric functions with respect to their indices would be involved, Eq. (6) for the location of the transition point \( k_c \) is given by a somewhat cumbersome transcendental equation. Instead of treating that, we first derive the MFPT using Eqs. (20) and (21) for \( r > 0 \) as follows,
\[
\langle T_r \rangle = q_0(x_0, s)|_{s=0} = \left[1 - \frac{1}{G(x_0, s = 0)}\right].
\]
(22)

In Fig. 1(b) we have plotted \( \langle T_r \rangle \) against \( r \) following Eq. (22).
FIG. 1. We show the variation of MFPT with resetting rate \( r \) for different \( k \), in cases of (a) linear, (b) harmonic, and (c) box potentials. We used \( x_0 = 0.5, D = 0.5 \) in the exact expressions for MFPT derived in the text. As may be seen, for \( k \gg k_c \), ORR \( r_e = 0 \).

\[
r = r_e(k) > 0 \text{ (ORR). For } k \gg k_c, \text{ ORR is zero. Beyond this we proceed numerically. We find the value of } r_e(k) \text{ within an accuracy of } 10^{-6}, \text{ and plot its rescaled dimensionless counterpart } z^2 \text{ against dimensionless potential strength } K = x_0 \sqrt{K/D} \text{ in Fig. 2(b). This helps us locate } k_c \text{ (and } K_c \simeq 0.7393) \text{ numerically. In Fig. 2(c) we plot } z^2 \text{ vs } (K_c - K) \text{ in a log-log scale for data values of } K \text{ very close to } K_c. \text{ The expected power law with power } \beta = 1 \text{ is shown.}
\]

C. Box potential \((n = \infty)\)

If we write \( k = k_0/L^{\alpha} \), then the potential \( V = k_0 (\frac{x}{L})^{\alpha} \). Now taking the limit \( n \to \infty \), we have \( V = 0 \) for \( x \ll L \) and \( V = \infty \) for \( x > L \), which is a box potential. In this limit, the modified form of the dimensionless potential strength is \( \lim_{n \to \infty} K = (\frac{k_0}{(\frac{L}{x_0})^{\alpha} \frac{L}{x_0} = (\frac{k_0}{L})^{\alpha} \frac{L}{x_0} = \frac{k_0}{L} \). As \( L \) becomes smaller, the strength \( K \) rises, and the diffusing particle is more effectively confined and assisted towards the capture site \( x = 0 \). The analog of Eq. (2) for this case is

\[
\frac{d^2 y}{dx^2} - \alpha^2 y = 0,
\]

with the boundary conditions \( y(0, s) = -\frac{r_0^0 + 1}{r + s} \) and \( \partial y / \partial x |_{x=L} = 0 \). The general solution is \( y = A_3 e^{\alpha x} + B_3 e^{-\alpha x} \), where \( A_3 \) and \( B_3 \) are fixed using the boundary conditions. This leads to

\[
q(x, s) = \left( \frac{r_0 + 1}{r + s} \right) - \frac{\cosh(\alpha(L - x))}{\cosh(\alpha L)} \right). \tag{24}
\]

From Eq. (24) or otherwise, without SR, \( F(x_0, s) = \cosh \left( \sqrt{\frac{D}{\alpha L}}(L - x_0) \right)/\cosh \left( \sqrt{\frac{D}{\alpha L}} \right) \). The latter implies that \( \langle T \rangle = -\left. \frac{dF}{ds} \right|_{x=0} = \frac{3}{2} \left( 2L - x_0 \right) \), and \( \sigma^2 = \langle T^2 \rangle - \langle T \rangle^2 = \frac{6 - (L - x_0)^2}{60D^2} \). Substituting these in Eq. (6), we see that the transition value of the potential is given by

\[
5K^2 - 10K_c + 4 = 0 \Rightarrow K_c = 1 - \frac{1}{\sqrt{5}}. \tag{25}
\]

The other root in Eq. (25) is ignored as \( K_c \leq 1 \).

The MFPT in this problem with SR is obtained from Eq. (24) as follows,

\[
\langle T_r \rangle = q_0(x_0, s)|_{s=0} = \frac{1}{r} \left[ \frac{\cosh \left( \sqrt{\frac{D}{\alpha L}} \right)}{\cosh \left( \sqrt{\frac{D}{\alpha L}}(L - x_0) \right)} - 1 \right]. \tag{26}
\]

In Fig. 1(c) we plot \( \langle T_r \rangle \) and see that the minimum at \( r = r_e(k) > 0 \) for \( L > L_e \) vanishes for \( L \leq L_e \). The exact expression for the \( r_e \) (ORR for \( K < K_c \)) is given by \( d\langle T_r \rangle / dr |_{r=r_e} = 0 \), which leads to the following transcendental equation,

\[
\left[ L \sinh \left( \sqrt{\frac{r_e}{D}} \right) - (L - x_0) \tanh \left( \sqrt{\frac{r_e}{D}}(L - x_0) \right) \cosh \left( \sqrt{\frac{r_e}{D}} \right) \right]
\]

\[
\approx 2 \sqrt{\frac{D}{r_e}} \left[ \cosh \left( \sqrt{\frac{r_e}{D}} \right) - \cosh \left( \sqrt{\frac{r_e}{D}}(L - x_0) \right) \right]. \tag{27}
\]

In Fig. 2(d) we plot \( z^2 \) against \( K \) (see the solid line) following Eq. (27). The ORR vanishes at \( K = K_c \) given by Eq. (25). In the limit of \( K \to K_c \), Eq. (5) yields \( r_e \), using the moments \( \langle T \rangle, \langle T^2 \rangle \), and \( \langle T^3 \rangle = \langle T \rangle^2 \langle 61L^4 - 14L^2(L - x_0)^2 + (L - x_0)^4 \rangle / 60D^2 \). This leads to

\[
z^2 = \frac{75(3\sqrt{5} - 5)}{4}(K_c - K). \tag{28}
\]
The above exact linear form, indicative of exponent $\beta = 1$, is shown as a dashed line in Fig. 2(d).

IV. ANALYTICAL RESULTS FOR ORR TRANSITION WITH STOCHASTIC TIME OVERHEAD

In many stochastic processes with resetting, there may be a finite refractory period (with a mean $\langle T_{\text{on}} \rangle$), as was discussed in the context of MMRS [11]. In this section, we discuss how ORR vanishes on varying the potential strength $K$, for $\langle T_{\text{on}} \rangle \neq 0$. The mean first passage time is given by Eq. (8). The ORR is obtained by the condition $d(T_r)/dr|_{r=r_0} = 0$ which gives [11]

$$r_0(1 + r_0(T_{\text{on}})) \frac{\partial \tilde{F}(r)}{\partial r} \bigg|_{r=r_0} = \tilde{F}(r_0)[\tilde{F}(r_0) - 1]. \tag{29}$$

Furthermore, the ORR vanishing is studied by Eq. (10). Thus apart from the expression of $r_0$ for $K < K_c$ [given by Eq. (29)], in the following we find the exact expressions of $K_c$ and small expansions of $r_0$ in terms of $(K_c - K)$ [using Eqs. (10) and (9)], for $n = 1$ and $n = \infty$.

For the linear potential ($n = 1$), using $\tilde{F}(r) = e^{\frac{1}{2} - (\frac{1}{2} + \frac{1}{\sqrt{2}})r_0}$ from Sec. III A, in Eq. (29) we find $r_0$ and hence $z^{2*}$ as a function of $K$. A plot of this is shown in Fig. 3(a) (solid line) for $\langle T_{\text{on}} \rangle = 0.1$. Then, substituting the necessary moments (from Sec. III A) in Eq. (10), we find the exact transition point,

$$K_c = \frac{4}{1 + \sqrt{1 + 16D\langle T_{\text{on}} \rangle/x_0^2}}. \tag{30}$$

which now depends on $\langle T_{\text{on}} \rangle$ and is $< 2$ for any $\langle T_{\text{on}} \rangle > 0$. Using the moments again in Eq. (9), we have

$$z^{2*} = \left[ \frac{3(x_0^4 + 4Dx_0^3K_c\langle T_{\text{on}} \rangle)(K_c - K)}{2(x_0^4 + 6Dx_0^3K_c\langle T_{\text{on}} \rangle + 6D^2K_c^2\langle T_{\text{on}} \rangle^2)} \right]^\frac{1}{2}, \tag{31}$$

for small $r_0$ near $K_c$, indicating $\beta = 1$.

Similarly for the box potential ($n = \infty$), using the function $\tilde{F}(r) = \cosh \left[ \frac{\sqrt{D}}{L}(L - x_0) \right]/\cosh \left( \frac{\sqrt{D}}{L}L \right)$ (from Sec. III C) in Eq. (29) we may obtain $z^{2*}$ vs $K$. For $\langle T_{\text{on}} \rangle = 0.1$ a plot of this is shown in Fig. 3(b) (solid line). Again, the relevant moments (from Sec. III C) substituted in Eq. (10) give

$$K_c = \frac{4}{5 + \sqrt{5 + 48D\langle T_{\text{on}} \rangle/x_0^2}}, \tag{32}$$

which is $< (1 - 1/\sqrt{5})$ for any $\langle T_{\text{on}} \rangle > 0$. Moreover, as $K \rightarrow K_c$, Eq. (9) gives the linear form (with $\beta = 1$) for

$$z^{2*} = \left[ \frac{30x_0^4K_c(4 - 5K_c)(K_c - K)}{x_0^4(11(1 - K_c)^2 + 6(1 - K_c)^2 - 1) - 72D^2\langle T_{\text{on}} \rangle^2K_c^4} \right]. \tag{33}$$

which is shown as a dashed line in Fig. 3(b).

V. NUMERICAL STUDY OF ORR TRANSITION IN GENERAL POTENTIAL $\langle x \rangle$

Since analytical solutions are often difficult to find in the case of arbitrary potentials $\langle x \rangle$, here we develop a numerical method to study the problem of the ORR transition in such situations. The aim will be to obtain $\langle T_r \rangle$ numerically first as a function of $r$, and then locate its minimum ($r_0$) for a given potential strength. Then one may vary the potential strength and study the corresponding variation and vanishing of $r_0$. While $\langle T_r \rangle$ may be obtained using kinetic Monte Carlo simulations [41], that would typically have relatively high statistical fluctuations, instead here we use a technique which is independent of statistical fluctuations.

We note that $\langle T_r \rangle = q(x_0, s)|_{s=0}$. In Sec. II, we discussed that the Laplace transform of the survival probability $q(x_0, s)$ is related to another function, $y(x_0, s) = q(x_0, s) + y(0, s)$, where $y(0, s) = -\frac{q(x_0, s)}{q(x_0, s)+1}$. One knows the differential equation satisfied by $y(x, s)$ (namely, Eq. (2) for $V(x) = k\langle x \rangle$) but its numerical solution is not straightforward, since its boundary condition $y(0, s)$ actually depends on the unknown $q(x_0, s)$ which we seek to find. This problem is avoided by studying instead a scaled function, $\tilde{y}(x, s) = y(x, s)/y(0, s)$, which has simpler boundary conditions $\tilde{y}(0, s) = 1$ and $\tilde{y}(\infty, s) = 0$ and satisfies the following equation,

$$\frac{d^2\tilde{y}}{dx^2} = \frac{V(x)}{D} \frac{d\tilde{y}}{dx} + \frac{(r + s)}{D} \tilde{y}. \tag{34}$$

We solve this differential Eq. (34) using the NDSOLVE technique in MATHEMATICA which includes the EXPLICITRUNGEKUTTA method to obtain $\tilde{y}(x, s)$. Since $q(x_0, s) = [\tilde{y}(x_0, s) - 1]y(0, s)$, we obtain

$$q(x_0, s) = \frac{1 - \tilde{y}(x_0, s)}{s + r\tilde{y}(x_0, s)}. \tag{35}$$

Thus the knowledge of the numerically determined $\tilde{y}(x_0, s)|_{s=0}$ finally gives us the desired mean first passage time,

$$\langle T_r \rangle = q(x_0, s)|_{s=0} = \frac{1 - \tilde{y}(x_0, 0)}{r\tilde{y}(x_0, 0)} \tag{36}$$

Sturm’s theorem discussed in Sec. II ensures that $y(x_0, 0)$ [and hence $\tilde{y}(x_0, 0)$] is nonzero for finite $x_0$, and Eq. (36) is therefore well defined. We checked the reliability of this technique by matching the exactly known $\langle T_r \rangle$ for linear and harmonic potentials from Eqs. (15) and (22)] to the numerically obtained $\langle T_r \rangle$ up to an accuracy of $10^{-8}$. This precision of $\langle T_r \rangle$ corresponded to our choice of a discrete step size of $10^{-4}$ for the variation of the resetting rate $r$. Thus all our answers below for values of $r_0$ are limited by this...
The numerical method. The numerical case, with precision. We apply the numerical method below to study a few different potentials.

A. Cubic ($kx^3$) and quartic ($kx^4$) potentials

For a chosen initial point $x_0 = 0.5$ and diffusion constant $D = 0.5$, we obtain $(T_r)$ as a function of $r$, and the corresponding optimal point $r_\ast$ for a given $k$. Note that although $r_\ast$ and the value of $k = k_c$ where $r_\ast$ vanishes depends on $x_0$ and $D$, we express our results in terms dimensionless quantities $z^2$ and $K$ which are supposed to be universal. For the cubic case, with $K = (\frac{1}{12})^3 x_0$, we find that $K_c = 0.6006 \pm 0.0001$. In Fig. 4(a), we plot $z^2$ against $(K_c − K)$ in a log-log plot and find a linear form valid over a couple of decades. For the quartic potential, with $K = (\frac{1}{24})^3 x_0$, we find that $K_c = 0.5597 \pm 0.0001$. In Fig. 4(b), we plot $z^2$ against $(K_c − K)$ in a log-log plot and find a linear form confirming again that $\beta = 1$.

All the analytical and numerical results for various powers $n$ of the power-law potential $V(x)$ may be summarized in Table I. We see that $K_c(n)$ decreases with $n$ and saturates as $n \to \infty$.

B. A nonmonotonic quartic potential

Here, we study $(T_r)$ vs $r$ for a one-dimensional potential whose behavior is somewhat different from the ones studied so far. The potential $V(x) = −\frac{b}{2} + \frac{1}{2}(1 − x^2)^2$ has a minimum at $x = x_{\text{min}} = 1$ and is plotted in Fig. 5(a) for various values of the parameter $b$. By increasing $b$, one may increase the depth $\Delta V = b/2$ of the potential. For $x > x_{\text{min}}$, the potential being attractive helps the Brownian particle reach the absorbing site ($x = 0$), while for $x < x_{\text{min}}$, a barrier resists an approach towards $x = 0$. Chemical reactions are often visualized as barrier crossing processes where the rate of reaction is the rate of first passage over the barrier. The current potential is motivated by such processes. We are interested how SR can influence such a barrier crossing.

In this problem we vary the parameter $b$ so as to simultaneously increase the depth of the potential well, and also make it steeper for $x > x_{\text{min}}$. For $b = 0$, to start with, we have a finite ORR ($r_\ast$) where $(T_r)$ has a minimum, for an initial location and reset point $x_0 = x_{\text{min}}$. The curiosity is to see that for the same $x_0$, with increasing $b > 0$, whether the ORR vanishes at some cutoff depth $b = b_c$, even when a barrier is present. Also, for any fixed $b$, we study how the ORR vanishing behaves as a function of $x_0$.

Using the numerical method discussed in this section, we find $(T_r)$ for this problem with $x_0 = 1$ and $D = 1$, and find that $r_\ast$ reduces with increasing $b$ and vanishes for $b \geq b_c = 0.3392 \pm 0.0001$. Thus even in the presence of a resisting barrier, the rising steepness of the potential (for $x > x_{\text{min}}$) supersedes its effect, and makes the advantage of resetting redundant beyond a point. In Fig. 5(b), we plot $r_\ast$ vs ($b_c − b$) in a log-log scale, and find a linear behavior indicating the exponent $\beta = 1$. For $b > b_c$, although there is no resetting advantage if the reset point is $x_0 = 1$, if one has a reset to a nearer point to the absorbing site, there is still an advantage. Similarly for $b < b_c$, the resetting advantage persists up to reset points $x_0 > 1$. These are quantitatively shown in Fig. 6—we see that ORR vanishing happens at $x_{0c} < 1$ for $b > b_c$ and at $x_{0c} > 1$ for $b < b_c$.

![Figure 4](image4.png)

**FIG. 4.** We show $z^2$ vs $K$ for cubic power and quadratic potentials. The numerical $K_c$ values are respectively $0.6006 \pm 0.0001$ and $0.5597 \pm 0.0001$.

![Figure 5](image5.png)

**FIG. 5.** We show in (a) the variation $V(x)$ with $x$ for various values of $b$. The curves are for $b = 0.1696$, $b_c = 0.3392$, and $b = 0.5088$. In (b) a log-log graph is shown for $r_\ast$ vs $b_c − b$. Here, $D = 1$ and $x_0 = 1$. A line with power 1 is in good agreement with the data.

![Figure 6](image6.png)

**FIG. 6.** We show $r_\ast$ vs $x_0$ for different values of $b = 0.1696$, $b_c = 0.3392$, and $b = 0.5088$. The corresponding $x_{0c}$ values are $1.06$, $1.0$, and $0.96$, respectively.

<table>
<thead>
<tr>
<th>Power $n$</th>
<th>$K_c = (\frac{1}{12})^3 x_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>$0.7393 \pm 0.0001$</td>
</tr>
<tr>
<td>3</td>
<td>$0.6006 \pm 0.0001$</td>
</tr>
<tr>
<td>4</td>
<td>$0.5597 \pm 0.0001$</td>
</tr>
<tr>
<td>$\infty$</td>
<td>$\frac{a}{b} = 1 - \frac{1}{c^3} \approx 0.5528$</td>
</tr>
</tbody>
</table>

**TABLE I.** The table contains $K_c$ with power $n$. 
VI. DISCUSSION

A variety of first passage processes in nature are guided by external fields. Examples include bacterial cells performing run and tumble in the presence of a spatial gradient of nutrients [42], or the movement of spindle microtubules towards the chromosomes guided by a biochemical gradient [43]. Such examples serve as a motivation for studying stochastic processes under bias. On the other hand, SR is known to serve as an optimal strategy for first capture processes. In this paper, we have studied a few toy examples in which the external bias competes with the events of SR. We have obtained conditions under which the advantage achieved by SR is annulled by the growing strength of the bias.

Here, we have studied it systematically as a function of varying strength of a potential that competes with SR at a constant rate. For a sufficiently strong potential, i.e., \( k \gg k_r \), the ORR vanishes. We derived the condition of the transition, dependent on the moments of first passage time without resetting [Eq. (6)]. Thus as the potential grows stronger and drives the particle more efficiently towards the capture site at the origin, the fluctuations in the first passage time characterized by \( \sigma^2 \) decrease until it matches the square of MFPT \( \langle T \rangle^2 \). Beyond that point, SR ceases to be of any extra assistance to the capture process. For processes with reset followed by a stochastic time overhead, we show that the condition of the transition is generalized to Eq. (10). Thus there is a limit to the advantage in first capture through SR, which is set by the degree to which a system is biased towards capture by an external force.

Related to the general results discussed above, we have several specific observations. The nondimensional critical potential strength \( K_c \) varies monotonically with \( n \) and reaches a constant value as \( n \to \infty \) (Table I, Sec. V). Also, we observed that in the presence of a finite refractory period \((\langle T_{\text{rn}} \rangle > 0)\), the values of \( K_c \) reduce in comparison to the cases with zero refractory period (Sec. IV). The exponent associated with the power-law form of vanishing \( r_s \) appears quite universally to be \( \beta = 1 \), owing to the ubiquitous analytic dependence of \( r_s \) on \( K \). We derive explicit analytic forms, both in the absence and presence of the refractory period, for \( n = 1 \) and \( n = \infty \). For \( n = 2 \) our analysis is mostly analytical. For \( n = 3 \) and 4 we obtain the results by numerically solving the relevant differential equations to a high degree of accuracy. The numerical method is applicable to any \( n \) and in fact to any arbitrary potential whose first derivative exists. As an example, we studied a nonmonotonic potential with a barrier near the origin. We find that as a function of the depth of the potential, the ORR vanishes at and above a critical depth and the associated exponent \( \beta = 1 \).

We believe that the transition studied in this paper and its mathematical criteria would be of general interest. The numerical method that we developed could be useful in studying other similar problems related to first capture processes.

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APPENDIX: DETAILED DERIVATION OF EQ. (4)

Starting from Eq. (3), if we Taylor expand the function \( \tilde{F}(r) \) about point \( r = 0 \), we have

\[
\langle T \rangle = \frac{1}{r} \left[ 1 - \frac{\tilde{F}(0) + r \frac{\partial \tilde{F}}{\partial r} \bigg|_{r=0} + \frac{r^2}{2!} \frac{\partial^2 \tilde{F}}{\partial r^2} \bigg|_{r=0} + \frac{r^3}{3!} \frac{\partial^3 \tilde{F}}{\partial r^3} \bigg|_{r=0} + O(r^4) \cdots }{\tilde{F}(0) + r \frac{\partial \tilde{F}}{\partial r} \bigg|_{r=0} + \frac{r^2}{2!} \frac{\partial^2 \tilde{F}}{\partial r^2} \bigg|_{r=0} + r^3 \frac{\partial^3 \tilde{F}}{\partial r^3} \bigg|_{r=0} + O(r^4) \cdots } \right].
\]  

(A1)

Here \( \tilde{F}(0) = 1 \), \( \langle T \rangle - \frac{\partial \tilde{F}}{\partial r} \bigg|_{r=0} = \langle T^2 \rangle = \frac{\partial^2 \tilde{F}}{2! \partial r^2} \bigg|_{r=0} \), and so on. Hence in terms of the different moments we may rewrite the MFPT as

\[
\langle T \rangle = \left[ \langle T \rangle - \frac{\langle T^2 \rangle}{2} + \frac{\langle T^3 \rangle}{3} - \frac{\langle T^4 \rangle}{4} + O(r^4) \cdots \right] \frac{1}{1 - r \langle T \rangle + \frac{\langle T^2 \rangle}{2} - \frac{\langle T^3 \rangle}{3} + O(r^4) \cdots}.
\]  

(A2)

A binomial expansion of the denominator in Eq. (A2) leads to

\[
\langle T \rangle = \langle T \rangle - r \left( \frac{\langle T^2 \rangle}{2} - \langle T \rangle^2 \right) + r^2 \left( \frac{1}{6} \langle T^3 \rangle + \langle T^3 \rangle - \langle T \rangle \langle T^2 \rangle \right) + r^3 \left( -\frac{\langle T^4 \rangle}{4!} + \frac{\langle T^3 \rangle \langle T \rangle}{3} + \frac{\langle T^2 \rangle^2}{4} - \frac{3 \langle T^2 \rangle \langle T^2 \rangle}{2} + \langle T^4 \rangle \right) + O(r^4) \cdots.
\]  

(A3)

To obtain the optimal resetting rate \( r_s \), we take a derivative of Eq. (A3) and set \( d\langle T_r \rangle/dr\bigg|_{r=r_s} = 0 \). This leads to a quadratic equation for \( r_s \)

\[
2cr_s^2 + 2br_s + a = 0,
\]  

(A4)

where \( a, b, \) and \( c \) are given by

\[
a = -\left( \frac{\sigma^2 - \langle T \rangle^2}{2} \right),
\]  

(A5)
\[ b = \left( \frac{1}{6} \langle T^3 \rangle - \sigma^2(T) \right), \tag{A6} \]
\[ c = 3 \left( -\frac{\langle T^4 \rangle}{4!} + \frac{\langle T^3 \rangle}{3} + \frac{\langle T^2 \rangle^2}{4} - \frac{3\langle T^2 \rangle}{2} + \langle T \rangle^4 \right). \tag{A7} \]

The solution of Eq. (A4) is \( r_s = \frac{-b \pm \sqrt{b^2 - 4ac}}{c}. \) In the immediate neighborhood of the transition point, as \( a \to 0, \) the value of \( r_s \) may be approximated as
\[ r_s \approx \frac{-b}{c} \pm \frac{b}{c} \left( 1 - \frac{ac}{2b^2} \right) = -\frac{a}{2b}, \tag{A8} \]

for the positive root.

The above equation is the same as Eq. (5) in the main text. Furthermore, Eq. (6) for the ORR transition point is obtained by setting \( a = 0. \)